1012. Proposed by George Apostolopoulos, Messolonghi, Greece.

Let ABC denote a triangle, I its incenter, s its semiperimeter, and $R_a, R_b,$ and R_c the circumradii of triangles IBC, ICA, and IAB, respectively. Prove that

(a) $\dfrac{a}{R_a} + \dfrac{b}{R_b} + \dfrac{c}{R_c} \le 3\sqrt{3}$, and

(b) $R_a + R_b + R_c \ge \dfrac{2s\sqrt{3}}{3}$

Solution by Arkady Alt , San Jose ,California, USA.

Since $\sin\angle BIC = \pi - \dfrac{B+C}{2} = \dfrac{\pi+A}{2}$ then applying Sin-Theorem to triangle IBC we obtain $\dfrac{BC}{\sin\angle BIC} = 2R_a \Leftrightarrow \dfrac{a}{\sin\dfrac{\pi+A}{2}} = 2R_a \Leftrightarrow R_a = \dfrac{a}{2\cos\dfrac{A}{2}} \Leftrightarrow R_a = 2R\sin\dfrac{A}{2}$.

Therefore, $\sum_{cyc}\dfrac{a}{R_a} \le 3\sqrt{3} \Leftrightarrow \boxed{\sum_{cyc}\cos\dfrac{A}{2} \le \dfrac{3\sqrt{3}}{2}}$ and, since $s = 4R\prod_{cyc}\cos\dfrac{A}{2}$, we obtain

$\sum_{cyc}R_a \ge \dfrac{2s\sqrt{3}}{3} \Leftrightarrow \sum_{cyc}2R\sin\dfrac{A}{2} \ge \dfrac{2}{\sqrt{3}}\cdot 4R\prod_{cyc}\cos\dfrac{A}{2} \Leftrightarrow \boxed{\sum_{cyc}\sin\dfrac{A}{2} \ge \dfrac{4}{\sqrt{3}}\prod_{cyc}\cos\dfrac{A}{2}}$.

Let $\alpha := \dfrac{\pi-A}{2}, \beta := \dfrac{\pi-B}{2}, \gamma := \dfrac{\pi-C}{2}$ then $\dfrac{A}{2} = \dfrac{\pi}{2} - \alpha, \dfrac{B}{2} = \dfrac{\pi}{2} - \beta, \dfrac{c}{2} = \dfrac{\pi}{2} - \gamma,$ where $\alpha,\beta,\gamma \in \left(0,\dfrac{\pi}{2}\right)$ and $\alpha+\beta+\gamma = \pi$.

Therefore, inequality (a) and (b), respectively, equivalent to

(1) $\sin\alpha + \sin\beta + \sin\gamma \le \dfrac{3\sqrt{3}}{2}$ and

(2) $\cos\alpha + \cos\beta + \cos\gamma \ge \dfrac{4}{\sqrt{3}}\sin\alpha\sin\beta\sin\gamma$.

Proof of (1).

Since $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2} \le 2\sin\dfrac{\alpha+\beta}{2} = 2\cos\dfrac{\gamma}{2}$ then

$\sin\alpha+\sin\beta+\sin\gamma \le 2\cos\dfrac{\gamma}{2} + 2\sin\dfrac{\gamma}{2}\cos\dfrac{\gamma}{2} = 2\cos\dfrac{\gamma}{2}\left(1+\sin\dfrac{\gamma}{2}\right) =$

$2\left(1+\sin\dfrac{\gamma}{2}\right)\sqrt{1-\sin^2\dfrac{\gamma}{2}} = 2\sqrt{(1-t)(1+t)^3}$, where $t = \sin\dfrac{\gamma}{2}$.

By AM-GM Inequality $(1-t)(1+t)^3 = \dfrac{(3-3t)(1+t)^3}{3} \le \dfrac{1}{3}\left(\dfrac{3-3t+3+3t}{4}\right)^4 = \dfrac{3^3}{2^4}$.

Hence, $\sin\alpha+\sin\beta+\sin\gamma \le 2\sqrt{\dfrac{3^3}{2^4}} = \dfrac{3\sqrt{3}}{2}$.

(Or another way to prove (1):

Since $\sin x$ is concave down on $(0,\pi)$ then, by Jensen Inequality, we obtain

$\dfrac{\sin\alpha+\sin\beta+\sin\gamma}{3} \le \sin\dfrac{\alpha+\beta+\gamma}{3} = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$).

Proof of (2).

Consider some triangle (acute) with angles α,β,γ. We will use regular notation for metric elements of this triangle, namely, let a,b,c be sidelengths of this triangle and s,R,r be it's semiperimeter, circumradius, inradius, respectively.

(don't mix them with notation of correspondent elements in original triangle).

Since $\cos\alpha+\cos\beta+\cos\gamma = 1 + \dfrac{r}{R}$ and $\sin\alpha\sin\beta\sin\gamma = \dfrac{abc}{8R^3} = \dfrac{4Rrs}{8R^3} = \dfrac{rs}{2R^2}$ then

(2)$\Leftrightarrow 1+\frac{r}{R} \geq \frac{4}{\sqrt{3}} \cdot \frac{rs}{2R^2} \Leftrightarrow 1+\frac{r}{R} \geq \frac{2rs}{R^2\sqrt{3}} \Leftrightarrow 2rs \leq \sqrt{3}R(R+r).$

Note that $\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\sqrt{3}}{2} \Leftrightarrow s \leq \frac{3\sqrt{3}}{2}R$. Since $2rs \leq 3\sqrt{3}Rr$ and $3\sqrt{3}Rr \leq \sqrt{3}R(R+r) \Leftrightarrow 3r \leq R+r \Leftrightarrow 2r \leq R$ (Euler Inequality) then $2rs \leq \sqrt{3}R(R+r)$ and that complete the proof.